

Robust Consensus Tracking of Heterogeneous Multi-Agent Systems under Switching Topologies

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Abstract In this paper, we consider a robust consensus tracking problem of heterogeneous multi-agent systems with time-varying interconnection topologies. Based on common Lyapunov function and internal model techniques, both state and output feedback control laws are derived to solve this problem. The proposed design is robust by admitting some parameter uncertainties in the multi-agent system.

Keywords Robust consensus tracking, multi-agent system, internal model, switching topology.

1 Introduction

The past decade has witnessed a rapid development in the field of multi-agent system and many fruitful results have been obtained. As one important topic, the consensus problem is actively studied due to its numerous applications such as cooperative control of unmanned aerial vehicles, communication among sensor networks, and formation of mobile robots (see [1, 2] and the references therein). Consensus means that a group of agents reaches an agreement on a physical quantity of interest by interacting with their local neighbors. Roughly speaking, existing consensus problems can be categorized into two types: consensus without a leader (i.e., leaderless consensus or synchronization) and consensus with a leader. The case of consensus with a leader is also called leader-following consensus or consensus tracking.

The consensus tracking problem of multi-agent systems has been studied for many years. This problem with agents in the form of single-integrators or double-integrators has been widely investigated ([2–4]). In [5], the authors proposed a distributed observer-based control law using local information to track an integrator-type leader. Later, this work has been extended to multi-agent systems with a general linear dynamics ([6, 7]) under switching topologies. Other extensions including consensus tracking with an unknown-input driven leader or a designed abstracted system have been studied in [8, 9, 22]

Recently, a general framework based on output regulation theory ([10]) has been developed for multi-agent consensus ([11, 12]), where distributed control could achieve both asymptotic tracking and disturbance rejection. For example, as an extension to the results in [6], a cooperative output regulation problem of heterogeneous multi-agent systems was formulated in [14] and solved by devising a distributed observer, where the exosystem can be viewed as the leader in the leader-following formulation. However, these results were obtained with a fixed topology. A similar problem but under a class of switching topologies was considered in [15]. Both dynamic state feedback and measurement output feedback control laws were proposed by using the full information of the exosystem which may be undesirable sometimes. Without requiring exactly known system matrices, many robust results were further obtained which can handle parameter uncertainties in the multi-agent system. For example, the authors in [11] proposed a robust control law based on internal model design with a no-loop graph which admits parameter uncertainties

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in the system matrices. Later, the no-loop graph assumption was removed in [12]. These results were mainly achieved with fixed topology. Very few results were obtained on robust output regulation with switching topologies, except that a control design was investigated in [16] based on canonical internal models to deal with small parameter uncertainties. However, this method was only applicable to the square follower system and led to an output feedback control law.

Our main motivation is to investigate the robust consensus tracking problem of heterogeneous multi-agent systems under switching topologies, which may have parameter uncertainties in their system matrices. To facilitate the control goal, a distributed observer regarding switching topologies is at first constructed for each agent, and thus the robust consensus tracking problem can be solved by incorporating an internal model in the controller for each agent. Our main contributions are at least two-fold:

- We extend the conventional leader-following consensus (e.g., [2, 5]) to general linear multi-agent systems with parameter uncertainties. When there are no such uncertainties, these results are consistent with existing consensus and distributed output regulation results, e.g., [5, 14].
- We consider the distributed/cooperative robust output regulation problem under switching topologies, and robust control laws are proposed for heterogeneous multi-agent systems, while many existing regulation results were derived for only homogeneous multi-agent systems ([12]) or for fixed graph cases ([11, 14]).

Moreover, both output and state feedback control laws are proposed to solve the robust output regulation problem considered in [16], and these control laws are also applicable to the follower system whose inputs have different dimensions with its outputs.

The rest of this paper is organized as follows. In Section 2, the problem formulation is given. Main results are presented in Section 3, where two types of robust consensus control laws are proposed. Finally, simulations and our concluding remarks are presented at the end.

Notations: Let \mathbb{R}^n be the n -dimensional Euclidean space, $\mathbb{R}^{n \times m}$ be the set of $n \times m$ real matrices, $\mathbf{0}_{n \times m}$ represents an $n \times m$ zero matrix. $\mathbf{1}_n$ represents a column vector of n entries equal to 1. $\text{diag}\{b_1, \dots, b_n\}$ denotes an $n \times n$ diagonal matrix with diagonal elements b_i ($i = 1, \dots, n$); $\text{col}(a_1, \dots, a_n) = [a_1^T, \dots, a_n^T]^T$ for any column vectors a_i ($i = 1, \dots, n$), $\text{vec}(X) = \text{col}(X_1, \dots, X_n)$ with X_i as the i -th column of a n -column matrix X . A weighted directed graph (or weighted digraph) $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ is defined as follows, where $\mathcal{N} = \{1, \dots, n\}$ is the set of nodes, $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ is the set of edges, and $\mathcal{A} \in \mathbb{R}^{n \times n}$ is a weighted adjacency matrix ([19]). $(i, j) \in \mathcal{E}$ denotes an edge leaving from node i and entering node j . The weighted adjacency matrix of this digraph \mathcal{G} is described by $A = [a_{ij}]_{i,j=1,\dots,n}$, where $a_{ii} = 0$ and $a_{ij} \geq 0$ ($a_{ij} > 0$ if and only if there is an edge from agent j to agent i). A path in graph \mathcal{G} is an alternating sequence $i_1 e_1 i_2 e_2 \dots e_{k-1} i_k$ of nodes i_l and edges $e_m = (i_m, i_{m+1}) \in \mathcal{E}$ for $l = 1, 2, \dots, k$. If there exists a path from node i to node j then node i is said to be reachable from node j . The neighbor set of agent i is defined as $\mathcal{N}_i = \{j: (j, i) \in \mathcal{E}\}$ for $i = 1, \dots, n$. A graph is said to be undirected if $a_{ij} = a_{ji}$ ($i, j = 1, \dots, n$). The weighted Laplacian $L = [l_{ij}] \in \mathbb{R}^{n \times n}$ of graph \mathcal{G} is defined as $l_{ii} = \sum_{j \neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ ($j \neq i$).

2 Problem Formulation

Consider a group of $N + 1$ agents, and N of them are followers described by

$$\begin{aligned} \dot{x}_i &= (A_i + \Delta A_i)x_i + (B_i + \Delta B_i)u_i \\ y_i &= (C_i + \Delta C_i)x_i + (D_i + \Delta D_i)u_i, \quad i = 1, \dots, N \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{p_i}$, $y_i \in \mathbb{R}^q$ are the state, input, and output of agent i , respectively. $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times p_i}$, $C_i \in \mathbb{R}^{q \times n_i}$, $D_i \in \mathbb{R}^{q \times p_i}$ represent the nominal part of the plant, while $\Delta A_i, \Delta B_i, \Delta C_i, \Delta D_i$ represent the uncertain part. Without loss of generality, assume the triple (C_i, A_i, B_i) is controllable and observable.

Let $w_i = \text{vec} \left(\begin{bmatrix} \Delta A_i & \Delta B_i \\ \Delta C_i & \Delta D_i \end{bmatrix} \right)$ be the uncertain vector for agent i and denote $A_i(w) = A_i + \Delta A_i$, $B_i(w) = B_i + \Delta B_i$, $C_i(w) = C_i + \Delta C_i$, $D_i(w) = D_i + \Delta D_i$, $w = \text{col}(w_1, \dots, w_N)$.

The leader(node 0) is as

$$\dot{v} = Sv, \quad y_0 = -Fv \quad (2)$$

where $S \in \mathbb{R}^{m \times m}$, $F \in \mathbb{R}^{q \times m}$, and (F, S) is observable.

Associated with this multi-agent system, a dynamic digraph \mathcal{G} can be defined with the nodes $\mathcal{N} = \{0, 1, \dots, N\}$ to describe the communication topology, which may be switching. If the control u_i can get access to the information of agent j at time instant t , there is an arc (j, i) in the graph \mathcal{G} , i.e., $a_{ij} > 0$. Also note that $a_{0i} = 0$ for $i = 1, \dots, N$ since the leader won't receive any information from the followers. Denote the induced subgraph associated with all followers as $\bar{\mathcal{G}}$. The associated Laplacian of this digraph can be partitioned as

$$L = \left[\begin{array}{c|c} 0 & \mathbf{0}_{1 \times N} \\ \hline \hat{L} & H \end{array} \right]$$

where $\hat{L} \in \mathbb{R}^{N \times 1}$ and $H \in \mathbb{R}^{N \times N}$.

To achieve the coordination of the multi-agent system, node 0 in \mathcal{G} should be globally reachable, and therefore at least one agent in each component of \mathcal{G} is connected to the leader. Otherwise, the coordination between the agents and the leader cannot be achieved. We say a communication graph is connected if the leader (node 0) is reachable from any other node of \mathcal{G} and the induced subgraph $\bar{\mathcal{G}}$ is undirected. By Lemma 3 in [5], H is positive definite if the communication graph is connected. Denote its eigenvalues as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N > 0$.

In multi-agent systems, the connectivity graph \mathcal{G} may be time-varying. To describe the variable interconnection topology, we denote all possible communication graphs as $\mathcal{G}_1, \dots, \mathcal{G}_\kappa$, $\mathcal{P} = \{1, \dots, \kappa\}$, and define a switching signal $\sigma : [0, \infty) \rightarrow \mathcal{P}$, which is piece-wise constant defined on an infinite sequence of nonempty, bounded, and continuous time-intervals. Assume $t_{i+1} - t_i \geq \tau_0$, $\forall i$, where t_i is the i -th switching instant and $t_0 = 0$. Here τ_0 is often called the dwell-time. Therefore, \mathcal{N}_i and the connection weight a_{ij} ($i, j = 0, 1, \dots, N$) are time-varying. Moreover, the Laplacian $L_{\sigma(t)}$ and the matrix $H_{\sigma(t)}$ associated with the switching interconnection graph $\mathcal{G}_{\sigma(t)}$ are also time-varying (switched at t_i , $i = 0, 1, \dots$), though they are time-invariant in each interval $[t_i, t_{i+1})$.

The following assumption on the communication graph is made.

Assumption 1 *The graph is switching among a group of connected graphs with the leader as its root.*

The robust consensus tracking problem of this heterogeneous multi-agent system can be formulated as follows. *Given the multi-agent system composed of the follower (1), the leader (2) and its corresponding communication graph $\mathcal{G}_{\sigma(t)}$, find a robust control law such that, there exists an open neighborhood W of $w = 0$, for any initial condition $x_i(0) \in \mathbb{R}^{n_i}$, $v(0) \in \mathbb{R}^m$, $w \in W$, the consensus tracking goal is achieved, i.e., $e_i = y_i - y_0 \rightarrow 0$ for $i = 1, \dots, N$ as $t \rightarrow \infty$.*

Note that, the system of agents considered are all with general linear models and can be seen as extensions to the well-studied consensus tracking problem in existing results for integrator-type agents ([1, 2, 5]).

Another assumption is needed to solve this robust consensus tracking problem.

Assumption 2 *Let λ be any eigenvalue of S , it holds*

$$\text{rank} \begin{bmatrix} A_i - \lambda I & B_i \\ C_i & D_i \end{bmatrix} = n_i + q. \quad (3)$$

Remark 1 *This condition is often called the transmission zero condition ([20]) and is a key ingredient to solve the robust output regulation problem in centralized cases.*

Remark 2 *When all agents have the same nominal system matrices, i.e., $A_i = A, B_i = B, C_i = C, D_i = D$, a similar result have obtained in [12] under a fixed topology. Note that the interconnection in our formulation allows switching topologies, which renders this problem more challenging.*

3 Main Results

In this section, we give a robust design scheme based on internal model techniques.

Unlike in decentralized cases ([17]), we do not assume the availability of y_0 to all agents in our problem. An agent can get access to y_0 unless there is an arc from the leader to this agent. Since not all followers can directly get access to the leader's information, we first build a distributed observer for each agent as follows.

$$\dot{\eta}_i = S\eta_i - L_0 F \eta_{vi} \quad (4)$$

where $\eta_{vi} = \sum_{j=0}^N a_{ij}(t)(\eta_i - \eta_j)$, $\eta_0 = v$ and L_0 is a gain matrix to be selected.

To establish the performance of this distributed observer, the following lemma will be used.

Lemma 1 *Let P be a positive definite symmetric solution of the Lyapunov inequality*

$$PS + S^T P - 2F^T F < 0. \quad (5)$$

Take $L_0 = \mu P^{-1} F^T$, then under Assumption 1, there exist positive constants μ^ and c , such that, when $\mu \geq \mu^*$, it holds*

$$(S - \lambda_i^p L_0 F)^T P + P(S - \lambda_i^p L_0 F) \leq -cP \quad (6)$$

where $\lambda_i^p > 0$ for $i = 1, \dots, N$ are the eigenvalues of H_p ($p \in \mathcal{P}$).

Proof: Note that there are only finite graphs satisfying Assumption 1, the minimum eigenvalue of H_p for all p is well-defined and denoted as $\bar{\lambda} > 0$. Taking $\mu^* = \max\{\frac{1}{\bar{\lambda}}, 1\}$, we obtain

$$\begin{aligned} & (S - \lambda_i^p L_0 F)^T P + P(S - \lambda_i^p L_0 F) \\ &= S^T P + PS - 2\mu \lambda_i^p F^T F \\ &= S^T P + PS - 2F^T F - 2(\mu \lambda_i^p - 1)F^T F \\ &\leq S^T P + PS - 2F^T F \end{aligned}$$

Since $S^T P + PS - 2F^T F$ is negative definite, there exists a sufficiently small constant c satisfying the inequality (6). ■

The next lemma shows the convergence of (4) and thus guarantees that each agent can asymptotically get the state information of the leader.

Lemma 2 Under Assumption 1, there exists a constant matrix L_0 , such that, η_i will exponentially converge to v as t goes to infinity, in the sense of $\|\eta_i - v\| \leq c_0 e^{-\lambda_0 t}$ for some constants c_0 and λ_0 .

Proof: Let $\eta = \text{col}(\eta_1, \dots, \eta_N)$ and $\bar{\eta} = \eta - \mathbf{1} \otimes v$. After some mathematical manipulations, it follows

$$\dot{\bar{\eta}} = (I_N \otimes S - H_{\sigma(t)} \otimes L_0 F) \bar{\eta}$$

Note that $H_{\sigma(t)}$ is constant and positive definite under Assumption 1 during each interval. We first consider this problem in an interval $[t_i, t_{i+1})$. Assume $\sigma(t) = p$ for $t \in [t_i, t_{i+1})$, there exists a unitary matrix U_p such that $\Lambda_p \triangleq U_p^T H_p U_p = \text{diag}\{\lambda_1^p, \dots, \lambda_N^p\}$. Let $\hat{\eta} = (U_p^T \otimes I_N) \bar{\eta}$, then,

$$\dot{\hat{\eta}} = (I_N \otimes S - \Lambda_p \otimes L_0 F) \hat{\eta}$$

that is, $\dot{\hat{\eta}}_i = (S - \lambda_i^p L_0 F) \hat{\eta}_i$ for $i = 1, \dots, N$. By selecting L_0 as that in Lemma 1 and letting $V_{\eta_i} = \hat{\eta}_i^T P \hat{\eta}_i$, we can derive $\dot{V}_{\eta_i} \leq -c V_{\eta_i}$. Recalling the dwell-time assumption, this inequality holds for all t . Let $V_{\eta}(t) = \sum_{i=1}^N V_{\eta_i}(t)$, it follows $\dot{V}_{\eta} \leq -c V_{\eta}$. Note that $\bar{\eta}^T \bar{\eta} = \hat{\eta}^T \hat{\eta}$, then

$$\|\eta_i - v\|^2 \leq \bar{\eta}^T \bar{\eta} \leq \lambda_{\min}(P)^{-1} V_{\eta}(t) \leq \lambda_{\min}(P)^{-1} V_{\eta}(0) e^{-ct}.$$

The conclusion follows readily. ■

Remark 3 It has been proved that, a sufficient and necessary condition for the linear matrix inequality (5) is the detectability of (F, S) ([18]). Hence, the observability of the leader system is sufficient to build an exponentially convergent distributed observer of the form (4) under Assumption 1.

Remark 4 A similar result has been obtained in [14] but under a fixed topology. Extensions under switching topologies were also obtained in [15] but assuming S has no eigenvalues with positive real parts. Lemma 2 allows that S can have eigenvalues with positive real parts and hence includes unbounded references for this multi-agent system under switching topologies. Moreover, while full-information of the leader was used in [14, 15], i.e., $F = I_q$, we only needs the referenced output y_0 here and avoid the cases when the availability of the leader's full-information is not desirable.

Next, we first consider a dynamic state feedback control to achieve the robust consensus tracking goal and then propose an output feedback design.

Let $\alpha(s) = s^{p_m} + \alpha_1 s^{p_m-1} + \dots + \alpha_{p_m-1} s + \alpha_{p_m}$ be the minimal polynomial of S , $G_1 = I_q \otimes \beta$, and $G_2 = I_q \otimes \gamma$ with

$$\beta = \left[\begin{array}{c|c} 0 & I_{p_m-1} \\ \hline -\alpha_{p_m} & -\alpha_{p_m-1} \dots \alpha_1 \end{array} \right], \quad \gamma = \text{col}(\underbrace{0, 0, \dots, 0}_{p_m-1}, 1).$$

Evidently, the pair (G_1, G_2) is controllable. It is the so-called q -copy of S ([20]). By incorporating its q -copy of the leader, we propose a state feedback control law of the form

$$\begin{aligned} u_i &= K_{1i} x_i + K_{2i} \xi_i \\ \dot{\xi}_i &= G_1 \xi_i + G_2 (y_i + F \eta_i) \\ \dot{\eta}_i &= S \eta_i - L_0 F \eta_{vi} \end{aligned} \tag{7}$$

where G_1, G_2 are as defined and K_{1i}, K_{2i} to be determined later.

By a similar proof of Lemma 1.23 in [20], under Assumption 2, the pair

$$\left(\left[\begin{array}{cc} A_i & \mathbf{0}_{n_i \times qp_m} \\ G_2 C_i & G_1 \end{array} \right], \left[\begin{array}{c} B_i \\ G_2 D_i \end{array} \right] \right) \tag{8}$$

is controllable for $i = 1, \dots, N$. Hence, there exist constant matrices K_{1i} and K_{2i} with proper dimensions such that the matrix

$$A_{ci} \triangleq \begin{bmatrix} A_i + B_i K_{1i} & B_i K_{2i} \\ G_2(C_i + D_i K_{1i}) & G_1 + G_2 D_i K_{2i} \end{bmatrix} \quad (9)$$

is Hurwitz.

The regulator equations play a key role in the output regulation problem ([20]), and the following lemma establishes its solvability for the coordination design. Its proof is similar with that of Lemma 1.27 in [20] and thus omitted.

Lemma 3 *There exists a neighborhood W_i of $w_i = 0$, such that for any $w_i \in W_i$, the following equations:*

$$X_i(w)S = A_i(w)X_i(w) + B_i(w)U_i(w), \quad (10)$$

$$Z_i(w)S = G_1 Z_i(w) \quad (11)$$

have a solution $(X_i(w), Z_i(w))$ with $U_i(w) = K_{1i}X_i(w) + K_{2i}Z_i(w)$. Moreover, $X_i(w)$ and $Z_i(w)$ satisfy

$$C_i(w)X_i(w) + D_i(w)U_i(w) + F = 0.$$

It is time to give our first main result.

Theorem 1 *Under Assumptions 1 and 2, the robust consensus tracking problem of this multi-agent system can be solved by the control law (7) with selected matrices K_{1i} , K_{2i} , $i = 1, \dots, N$, and L_0 .*

Proof: Under Assumption 2 and by Lemma 3, for any i , there exists a neighborhood W_i of $w_i = 0$, such that for any $w_i \in W_i$, there exist two matrices $X_i(w)$ and $Z_i(w)$ satisfying

$$X_i(w)S = A_i(w)X_i(w) + B_i(w)U_i(w)$$

$$Z_i(w)S = G_1 Z_i(w)$$

where $U_i(w) = K_{1i}X_i(w) + K_{2i}Z_i(w)$. Taking W as the direct product of all W_i (i.e., $W = W_1 \times \dots \times W_N$) and performing a transformation $\bar{x}_i = x_i - X_i(w)v$, $\bar{\xi}_i = \xi_i - Z_i(w)v$ gives

$$\begin{aligned} \dot{\bar{x}}_i &= (A_i(w) + B_i(w)K_{1i})\bar{x}_i + B_i(w)K_{2i}\bar{\xi}_i \\ \dot{\bar{\xi}}_i &= G_2(C_i(w) + D_i(w)K_{1i})\bar{x}_i + (G_1 + G_2 D_i(w)K_{2i})\bar{\xi}_i - G_2 \bar{\eta}_i \\ e_i &= (C_i(w) + D_i(w)K_{1i})\bar{x}_i + D_i(w)K_{2i}\bar{\xi}_i \end{aligned}$$

Denote $\bar{x}_{ci} = \text{col}(\bar{x}_i, \bar{\xi}_i)$ and rewrite these equations in a compact form:

$$\begin{aligned} \dot{\bar{x}}_{ci} &= A_{ci}(w)\bar{x}_{ci} + E_{ci}(w)\bar{\eta}_i \\ e_i &= C_{ci}(w)\bar{x}_{ci} \end{aligned}$$

where $E_{ci}(w) = \text{col}(\mathbf{0}_{n_i \times n_z}, G_2 F)$, $C_{ci}(w) = [C_i(w) + D_i(w)K_{1i}, D_i(w)K_{2i}]$ are constant matrices with uncertain parameters. By the exponential stability of $A_{ci}(w)$ and Lemma 2, we apply Lemmas 4.6 and 4.7 in [21] and obtain the convergence of \bar{x}_{ci} . As a result, the proof is completed. \blacksquare

Remark 5 *It can be found that if $y_0 = v$ (i.e., $F = -I_q$), it means that the state variable v of the leader can be directly obtained when it is reachable from some other agent. This special case has been partly considered in [14, 15]. We extend the control law in [14, 15] by only using the measurement output of the leader. Similar control laws were proposed in [5, 6] when the followers are all integrators, while here we consider more general cases in this formulation.*

In some circumstances, the follower may not have access to its own state and only its output measurement is available. The following theorem shows how the robust consensus tracking problem can be solved by an output feedback control law.

Theorem 2 *Under Assumptions 1 and 2, there exist proper matrices K_{1i} , K_{2i} , K_{3i} , and L_0 , such that the robust consensus tracking problem of this multi-agent system is solved by the following control law:*

$$\begin{aligned} u_i &= K_{1i}\zeta_i + K_{2i}\xi_i \\ \dot{\zeta}_i &= A_i\zeta_i + B_i u_i + K_{3i}(y_i - C_i\zeta_i - D_i u_i) \\ \dot{\xi}_i &= G_1\xi_i + G_2(y_i + F\eta_i) \\ \dot{\eta}_i &= S\eta_i - L_0 F\eta_{vi} \end{aligned} \tag{12}$$

Proof: The proof is similar with that of Theorem 1. Since the pair (8) is controllable, we still take K_{1i} and K_{2i} as Theorem 1. Note that (C_i, A_i) is observable, there exists an K_{3i} such that $A_i - K_{3i}C_i$ is Hurwitz. Denote $\hat{x}_{ci} = \text{col}(x_i, \zeta_i, \xi_i)$ and consider the closed-loop system of agent i in a compact form as

$$\dot{\hat{x}}_{ci} = \bar{A}_{ci}(w)\hat{x}_{ci} + \bar{E}_{ci}(w)\eta_i$$

where $\bar{E}_{ci}(w) = \text{col}(\mathbf{0}_{n_i \times m}, \mathbf{0}_{n_i \times m}, G_2 F)$, and

$$\bar{A}_{ci}(w) = \begin{bmatrix} A_i(w) & B_i(w)K_{1i} & B_i(w)K_{2i} \\ K_{3i}C_i(w) & \Xi_{1i} & \Xi_{2i} \\ G_2C_i(w) & G_2D_i(w)K_{1i} & G_1 + G_2D_i(w)K_{2i} \end{bmatrix},$$

with $\Xi_{1i} \triangleq A_i - K_{3i}C_i(w) + B_iK_{1i} + K_{3i}(D_i(w) - D)K_{1i}$ and $\Xi_{2i} \triangleq B_iK_{2i} + K_{3i}(D_i(w) - D)K_{2i}$.

The nominal system matrix $\bar{A}_{ci}(0)$ is

$$\bar{A}_{ci}(0) = \begin{bmatrix} A_i & B_iK_{1i} & B_iK_{2i} \\ K_{3i}C_i & A_i - K_{3i}C_i + B_iK_{1i} & B_iK_{2i} \\ G_2C_i & G_2D_iK_{1i} & G_1 + G_2D_iK_{2i} \end{bmatrix},$$

Let

$$T_i = \begin{bmatrix} I_{n_i} & 0 & 0 \\ -I_{n_i} & I_{n_i} & 0 \\ 0 & 0 & I_{n_z} \end{bmatrix}.$$

Then,

$$\begin{aligned} \hat{A}_{ci}(0) &\triangleq T_i \bar{A}_{ci} T_i^{-1} \\ &= \begin{bmatrix} A_i + B_iK_{1i} & B_iK_{1i} & B_iK_{2i} \\ 0 & A_i - K_{3i}C_i & 0 \\ G_2(C_i + D_iK_{1i}) & G_2D_iK_{1i} & G_1 + G_2D_iK_{2i} \end{bmatrix} \end{aligned}$$

Recalling the selection of $K_{1i}, K_{2i}, K_{3i}, \hat{A}_{ci}(0)$ and hence \bar{A}_{ci} is Hurwitz. There exists a neighborhood W of $w = 0$, such that $\bar{A}_{ci}(w)$ is also Hurwitz. Following the same procedure as Lemma 3, there exist unique matrices $\bar{X}_i(w)$, $\bar{Y}_i(w)$, and $\bar{Z}_i(w)$ satisfying

$$\begin{aligned} \bar{X}_i(w)S &= A_i(w)\bar{X}_i(w) + B_i(w)\bar{U}_i(w) \\ \bar{Y}_i(w)S &= A_i(w)\bar{X}_i(w) + B_i(w)\bar{U}_i(w) + K_{3i}C_i(w)(\bar{X}_i(w) - \bar{Y}_i(w)) \\ \bar{Z}_i(w)S &= G_1\bar{Z}_i(w) + G_2[C_i(w)\bar{X}_i(w) + D_i(w)\bar{U}_i(w) + F] \end{aligned}$$

where $\bar{U}_i(w) = K_{1i}\bar{Y}_i(w) + K_{2i}\bar{Z}_i(w)$. From the uniqueness, $\bar{X}_i(w) = \bar{Y}_i(w)$. By the similar techniques of Lemma 1.27 in [20], we can also obtain $C_i(w)\bar{X}_i(w) + D_i(w)\bar{U}_i(w) + F = 0$.

Performing a transformation $\bar{x}_i = x_i - \bar{X}_i(w)v$, $\bar{\zeta}_i = \zeta_i - \bar{Y}_i(w)v$, $\bar{\xi}_i = \xi_i - \bar{Z}_i(w)v$ and denoting $\bar{x}_{ci} = \text{col}(\bar{x}_i, \bar{\zeta}_i, \bar{\xi}_i)$ gives

$$\begin{aligned}\dot{\bar{x}}_{ci} &= \bar{A}_{ci}(w)\bar{x}_{ci} + \bar{E}_{ci}(w)\bar{\eta}_i \\ e_i &= C_{ci}(w)\bar{x}_{ci}\end{aligned}$$

where $C_{ci}(w) = [C_i(w), D_i(w)K_{1i}, D_i(w)K_{2i}]$ are constant matrices with uncertainties. By the stability of $\bar{A}_{ci}(w)$ and Lemma 2, we apply Lemmas 4.6 and 4.7 in [21] and obtain the convergence of \bar{x}_{ci} . As a result, the proof is completed. \blacksquare

Remark 6 *This problem has been partially investigated in [15] under switching topologies. However, the constructed controllers heavily relied on the exact known system matrices and hence can't admit uncertainties. The internal model-based control law here facilitates us to allow small parameter uncertainties and thus obtain a robust control law. Moreover, without restricting the eigenvalues of S are in the closed left half-plane, the common Lyapunov function technique helps us handle the switching topologies even the references may be unbounded.*

Remark 7 *Another relevant paper is [16], in which a robust output regulation problem of linear multi-agent systems was studied by a so-called canonical internal model. The main differences between our work and that of [16] are at least two-fold. First, the method proposed in [16] only applies to the case when the outputs have the same dimension with its inputs. Second, only an output feedback control law was derived in [16], while here both state feedback and output feedback control laws are proposed. Moreover, since the out feedback control law in [16] needs to estimate the state of the internal model, its dimension may be much higher than the control (12) when $q > 1$.*

4 Simulations

For illustrations, we present an example and consider three followers with following system matrices

$$\begin{aligned}A_1 &= 1 + \epsilon_1, \quad B_1 = 1 + \epsilon_1, \quad C_1 = 1 + \epsilon_1, \quad D_1 = 1 + \epsilon_1, \\ A_2 &= \begin{bmatrix} 0 & 1 + \epsilon_2 \\ -1 + \epsilon_2 & 0 \end{bmatrix}, \quad B_2 = \text{col}(0, 1 + \epsilon_2), \quad C_2 = [1 + \epsilon_2, 0], \quad D_2 = \epsilon_2, \\ A_3 &= \begin{bmatrix} \epsilon_3 & 1 & 0 \\ -1 + \epsilon_3 & 0 & 1 \\ 2 & \epsilon_3 & 1 \end{bmatrix}, \quad B_3 = \text{col}(0, 1, 1 + \epsilon_1), \quad C_3 = [0, 1, 0], \quad D_3 = 1.\end{aligned}$$

The leader is described by

$$\dot{v}_1 = v_2, \quad \dot{v}_2 = -v_1, \quad y_0 = -v_1.$$

We assume the interconnection topology is switching between graph \mathcal{G}_i ($i = 1, 2$) described by Figure 1. The switchings are periodically carried out as $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_1, \mathcal{G}_2, \dots\}$ with periods $t = 5$.

The uncertain parameters $\epsilon_1, \epsilon_2, \epsilon_3$ are taken between $[-1, 1]$. The robust consensus tracking performances under state and output feedback control laws were depicted in Figures 2 and 3.

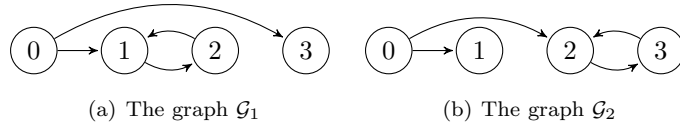


Figure 1: The communication graphs.

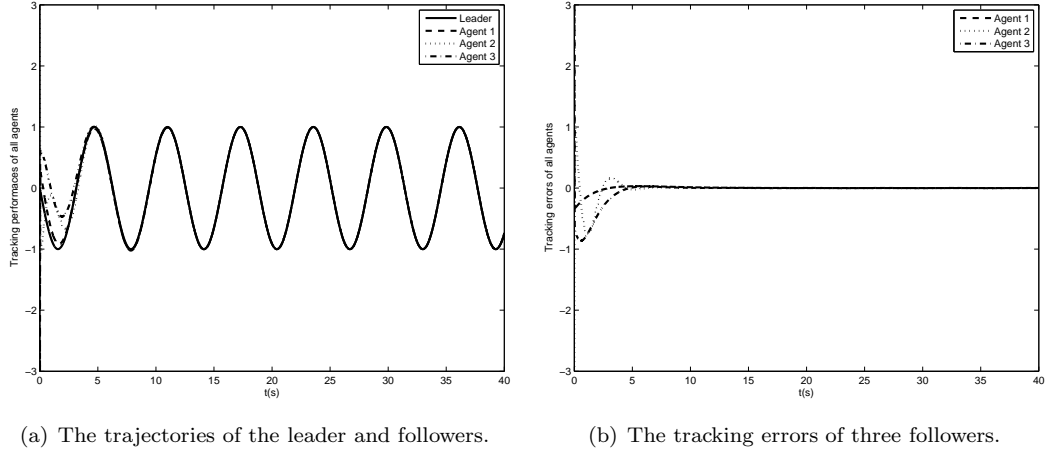


Figure 2: The tracking performance of three agents under the controller (7).

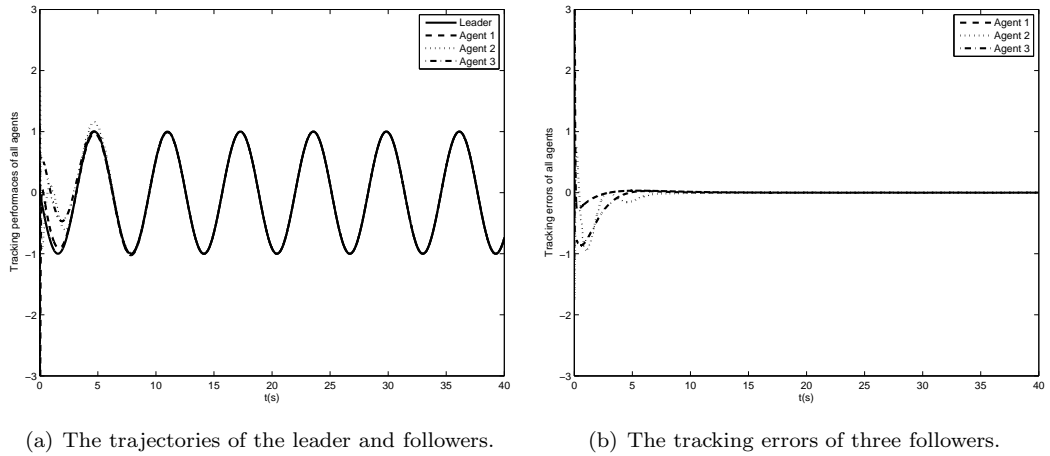


Figure 3: The tracking performance under the controller (12).

5 Conclusions

A robust consensus tracking problem of heterogeneous multi-agent systems was solved to admit small uncertain parameters in the agents' systems. In conjunction with the internal model techniques, a common Lyapunov function was used to overcome the challenges of switching topologies. Future work includes extensions to more general graphs and multiple leaders.

References

- [1] Fax J and Murray R, Information flow and cooperative control of vehicle formations. *IEEE Trans. Automat. Control*, 2004, **49**(9): 1465–1476.
- [2] Ren W and Beard R, *Distributed Consensus in Multi-Vehicle Cooperative Control: Theory and Applications*, Springer-Verlag, London, 2008.
- [3] Li L, Ho D, and Lu J, A unified approach to practical consensus with quantized data and time delay, *IEEE Trans. Circuits Syst. I. Regul. Pap.*, 2013, **60**(10): 2668–2678.
- [4] Meng Z, Ren W, Cao Y, and You Z, Leaderless and leader-following consensus with communication and input delays under a directed network topology, *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, 2011, **41**(1): 75–88.
- [5] Hong Y, Hu J, and Gao L, Tracking control for multi-agent consensus with an active leader and variable topology, *Automatica*, 2006, **42**(7): 1177–1182.
- [6] Hong Y and Wang X, Multi-agent tracking of a high-dimensional active leader with switching topology, *J. Syst. Sci. Complex.*, 2009, **22**(4): 722–731.
- [7] Ni W and Cheng D, Leader-following consensus of multi-agent systems under fixed and switching topologies, *Systems Control Lett.*, 2010, **59**(3): 209–217.
- [8] Tang Y, Leader-following coordination problem with an uncertain leader in a multi-agent system, *IET Control Theory Appl.*, 2014, **8**(10): 773–781.
- [9] Tang Y, Hong Y, Hierarchical distributed control design for multi-agent systems using approximate simulation. *Acta Automatica Sinica*, 2013, **39**(6): 868–874.
- [10] Wonham W, *Linear Multivariable Control: A Geometric Approach*, Springer-Verlag, New York, 1979.
- [11] Wang X, Hong Y, Huang J, and Jiang Z, A distributed control approach to a robust output regulation problem for multi-agent linear systems, *IEEE Trans. Automat. Control*, 2010, **55**(12): 2891–2895.
- [12] Su Y, Hong Y, and Huang J, A general result on the robust cooperative output regulation for linear uncertain multi-agent systems, *IEEE Trans. Automat. Control*, 2013, **58**(5): 1275–1279.
- [13] Tang Y, Output consensus of monlinear multi-agent systems with unknown control directions, *Kybernetika*, 2015, **51**(2): 335–346.
- [14] Su Y and Huang J, Cooperative output regulation of linear multi-agent systems, *IEEE Trans. Automat. Control*, 2012, **57**(4): 1062–1066.

- [15] Su Y and Huang J, Cooperative output regulation with application to multi-agent consensus under switching network, *IEEE Trans. Syst., Man, Cybern., Syst.*, 2012, **42**(3): 864–875.
- [16] Wang X and Han F, Robust coordination control of switching multi-agent systems via output regulation approach, *Kybernetika*, 2011, **47**(5): 755–772.
- [17] Xi Z and Ding Z, Global decentralised output regulation for a class of large-scale nonlinear systems with nonlinear exosystem. *IET Control Theory Appl.*, 2007, **1**(5): 1504–1511.
- [18] Boyd S, L. Ghaoui EI, Feron E, and Balakrishnan V, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, 1994.
- [19] Godsil C and Royle G, *Algebraic Graph Theory*, Springer-Verlag, New York, 2001.
- [20] Huang J, *Nonlinear Output Regulation: Theory and Applications*, SIAM, Philadelphia, 2004.
- [21] Khalil H, *Nonlinear Systems (3rd)*, Prentice Hall, New Jersey, 2001.
- [22] Tang Y, Hong Y, and Wang X, Distributed output regulation for a class of nonlinear multi-agent systems with unknown-input leaders, *Automatica*, to appear. Available at: <http://arxiv.org/abs/1508.00213>.